effect of a concentrated force along the interface, it again becomes elementary.

In conclusion, let us note the possibility of applying the method to solve the same problems for media with a more general kind of anisotropy.

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STRESS CONDITIONS IN PLATES REINFORCED BY STIFFENING RIBS

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The problem of sresses transmitted through a stiffening rib in a plate is usually examined under various simplifying assumptions (see e. g. [1–5]).

A sufficiently simple method is proposed below for effective construction of solutions for problems of this type. This approach based on known methods of solution of planar problems permits to construct the solution in finite form.

The solution is found in integrals of the Cauchy type. The density of these integrals is determined by means of Fourier transformation.

1. The method of solution will be presented using as an example an elastic half-plane reinforced by a semi-infinite straight stringer (stiffening rib) continuously attached to the plane along the boundary.

We shall assume that the stresses (in the plate and in the stringer) are produced by only one axial force applied at the end of the stringer.

We locate the plate in the lower half-plane of the plane of the complex variable $z = x + iy$ and let the stringer coincide with the positive part of the real axis. One end of the stringer will be at the origin of coordinates, the other will extend to infinity.

Let E and ν be the elastic constants of the plate, E_0 the modulus of elasticity of the rod, h the thickness of the plate, S_0 the cross section of the rod, assumed to be constant. The magnitude of the external force applied at the end of the rod and directed along the x-axis is designated through p_0. The remaining notations used below are commonly accepted.

The part of the boundary to the left of the origin of coordinates is by definition not stressed. Therefore the boundary conditions will be

$$\sigma_y = \tau_{xy} = 0, \qquad x < 0 \tag{1.1}$$

The boundary conditions on the remaining part of the boundary where the plate is connected to the stringer will consist of conditions of equilibrium of any part $(0, x)$ of the latter. These conditions are given by the following two equations:

$$p_0 - h\int_0^x \tau_{xy}\,dt + k\sigma_x = 0$$
$$-h\int_0^x \sigma_y\,dt = 0, \qquad x > 0; \qquad k = \frac{E_0 S_0}{E} \tag{1.2}$$

These two equations together give

$$p_0 - h\int_0^x (\tau_{xy} + i\sigma_y)\,dt + k\sigma_x = 0 \qquad (x > 0) \tag{1.3}$$

We shall take advantage of the well known formulations of Koslov-Muskhelishvili [6]

$$\sigma_x + \sigma_y = 2[\varphi'(z) + \overline{\varphi'(z)}] \tag{1.4}$$
$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\varphi''(z) + \psi'(z)]$$

and the equation of Muskhelishvili (same reference)

$$-i\int_0^t (\tau_{xy} + i\sigma_y)\,d\tau = \varphi(t) + \overline{t\varphi'(t)} + \overline{\psi(t)} + \text{const} \tag{1.5}$$

On the basis of these equations the boundary conditions (1.1) and (1.3) of the problem under examination are written in the following form after omission of insignificant constants:

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = 0 \qquad (t < 0) \tag{1.6}$$

$$ip_0 + h[\varphi(t) + t\overline{\varphi'(t)} \quad \overline{\psi(t)}] + \tag{1.7}$$
$$+ ik\mathrm{Re}[\varphi'(t) + \overline{\varphi'(t)} - \overline{t\varphi''(t)} - \overline{\psi'(t)}] = 0 \qquad (t > 0)$$

According to the physical meaning of the problem it is completely clear that the stresses will not be bounded in the closed half-plane. At the point $z = 0$ where the load acts on the plate through the stringer the stresses can go to infinity of the order lower than unity. For large $|z|$ the stresses will be vanishingly small in view of the absence of external stresses in the infinitely elongated parts of the medium.

Let us compute the principal vector of external stresses applied to the boundary of the half-plane. On the basis of (1.1) and (1.2) we have

$$X = \int_{-\infty}^{\infty} \tau_{xy}\,dt = \frac{p_0}{h}, \qquad Y = \int_{-\infty}^{\infty} \sigma_y\,dt = 0 \tag{1.8}$$

The presence of a principal vector different from zero has, according to expressions of Muskhelishvili, as a consequence the unboundedness of complex potentials $\varphi(z)$ and $\psi(z)$ at infinity. For large $|z|$ the potentials will have the form ([6], p. 346)

$$\varphi(z)=-\frac{X}{2\pi}\ln z+\varphi_0(z),\qquad \psi(z)=\frac{X}{2\pi}\ln z+\psi_0(z) \tag{1.9}$$

where $\varphi_0(z)$ and $\psi_0(z)$ are functions holomorphic in the half-plane, which permit the asymptotic representation

$$\varphi_0(z)=o(1)+\text{const},\qquad \psi_0(z)=o(1)+\text{const} \tag{1.10}$$

In accordance with the assumption about the character of stresses in the vicinity of the point $z=0$, the boundary values of function $\varphi'(z)$ can allow at this point a singularity of the order lower than unity.

Together with the lower half-plane, which we now designate by S^-, we also introduce into the analysis the upper half-plane $\operatorname{Im} z>0$, to be designated below by S^+. Following Muskhelishvili [6], we extend the determination of function $\varphi(z)$ to the upper half-plane, assuming that

$$\varphi(z)=-z\overline{\varphi}'(z)-\overline{\psi}(z)\quad\text{for } z \text{ in } S^+ \tag{1.11}$$

On the basis of the boundary value (1.6) it is easy to infer that function (1.11), which is holomorphic in S^+, analytically extends the values of the complex potential $\varphi(z)$ from S^- through the pole $(-\infty, 0)$.

The function determined in this manner which we again designate through $\varphi(z)$ will be piecewise holomorphic in the region S which represents the entire plane z bisected along the half-axis $(0, \infty)$. Through this function the function $\psi(z)$ can be expressed from (1.11) in the following form:

$$\psi(z)=-\overline{\varphi}(z)-z\varphi'(z)\quad\text{for } z \text{ in } S^- \tag{1.12}$$

Equations (1.11) and (1.12) allow the assertion that expression (1.9) is also valid for a piecewise holomorphic function $\varphi(z)$.

In the examination of the plane which is bisected along the positive part of the real axis we shall distinguish between the upper and lower edge of the cut and will assign to values related to these edges the signs plus and minus, respectively.

Functions $\psi(t)$ and $\psi'(t)$ (more correctly $\psi^-(t)$ and $\psi'^-(t)$) in boundary conditions (1.6) and (1.7) are replaced by their values (1.12). Then these conditions take the form

$$\varphi^-(t)-\varphi^+(t)=0,\qquad t<0 \tag{1.13}$$

$$ip_0+h\,[\varphi^-(t)-\varphi^+(t)]+ik\operatorname{Re}[\varphi'^+(t)+3\varphi'^-(t)]=0,\quad t>0$$

To solve this problem, we write in region S

$$\varphi(z)=-\frac{X}{2\pi}\ln z+\varphi_0(z),\qquad \varphi_0(z)=\frac{1}{2\pi i}\int_0^\infty\frac{\omega(\tau)\,d\tau}{\tau-z} \tag{1.14}$$

Here $\omega(\tau)=\mu(\tau)+i\nu(\tau)$ is a new unknown function on $[0,\infty)$, while $\ln z$ is understood to be a definite branch of this function which assumes, let us say, positive values on the upper edge of the cut.

With respect to function $\omega(\tau)$ we shall assume that it together with its derivative of the first order satisfies Hoelder's condition on any finite section which does not have as its end $\tau=0$. We shall also assume that $\omega(\tau)$ belongs to the class $L_p(0,\infty)$ for some $p>1$, and $\omega'(\tau)$ to the class $L(0,\infty)$. As a result we have

$$\varphi_0'(z) = -\frac{\omega(0)}{2\pi i z} + \frac{1}{2\pi i}\int_0^\infty \frac{\omega'(\tau)\,d\tau}{\tau - z}$$

$$\varphi_0^{\pm}(t) = \pm\frac{1}{2}\,\omega(t) + \frac{1}{2\pi i}\int_0^\infty \frac{\omega(\tau)\,d\tau}{\tau - t} \tag{1.15}$$

$$\varphi_0{}^{\pm}(t) = -\frac{\omega(0)}{2\pi i t} \pm \frac{1}{2}\,\omega'(t) + \frac{1}{2\pi i}\int_0^\infty \frac{\omega'(\tau)\,d\tau}{\tau - t} \qquad (t > 0)$$

If the preceding expressions for the boundary values of function $\varphi(z)$ and of its derivative are substituted into (1.13) and if it is taken into account here that

$$\varphi^-(t) - \varphi^+(t) = -\,\omega(t) - \frac{X}{2\pi}\,[\ln^- t - \ln^+(t)] = -\,\omega t - iX = -\,\omega(t) - \frac{ip_0}{h}$$

then we obtain

$$h\omega(t) + ik\left\{\frac{2\,[X + \nu(0)]}{\pi t} + \mu'(t) - \frac{2}{\pi}\int_0^\infty \frac{\nu'(\tau)\,d\tau}{\tau - t}\right\} = 0 \tag{1.16}$$

Let us try to determine the quantity $\nu(0)$. According to (1.4) we have

$$\tau_{xy}^- = \operatorname{Im}\,\{t\varphi''^-(t) + \psi'^-(t)\}$$

If the function $\psi'^-(t)$ is replaced here by its expression from (1.12), then the first part of the equation takes on the form

$$\operatorname{Im}\,\{t\varphi''^-(t) + \psi'^-(t)\} = -\operatorname{Im}\{\overline{\varphi'^+(t)} + \varphi'^-(t)\}$$

For limit values of function $\varphi'(z)$ in the preceding equation we take advantage of Eqs. (1.15) of Sakhotskii-Plemel. As a result we obtain

$$\tau_{xy}^- = \nu'(t) \tag{1.17}$$

Let us turn now to the first equation (1.2). By virtue of (1.17) it assumes the form

$$p_0 - h\,[\nu(x) - \nu(0)] + k\sigma_x = 0 \qquad (x > 0)$$

Here the transition to the limit for $x \to \infty$ determines $\nu(0)$ in the following form:

$$\nu(0) = -\frac{p_0}{h} \tag{1.18}$$

We note now that in view of (1.8) and (1.18) the first term in braces in (1.16) drops out. After separation of real and imaginary parts, Eq. (1.16) is presented in the form of two equations

$$\mu(t) = 0, \qquad \nu(t) - \frac{\lambda}{\pi}\int_0^\infty \frac{\nu'(\tau)\,d\tau}{\tau - t} = 0 \qquad \left(\lambda = \frac{2k}{h} = \frac{2E_0S_0}{Eh}\right) \tag{1.19}$$

In this manner a homogeneous integro-differential equation (1.19) is obtained for the determination of density of integral (1.14) under the condition (1.18).

Note. Equation (1.19) can be easily be given the form of an inhomogeneous singular integral equation. In fact, it follows from Eqs. (1.17) and (1.18) that:

$$\nu(t) = \int_0^t \tau_{xy}^-\,d\tau - \frac{p_0}{h}$$

Substituting these expressions together with (1.17) into (1.19). we find

$$\frac{\lambda}{\pi}\int_0^\infty \frac{\tau_{xy}^-\,d\tau}{\tau - t} - \int_0^t \tau_{xy}^-\,d\tau + \frac{p_0}{h} = 0 \tag{1.20}$$

Equation (1.20) was obtained from other considerations in the work of Koiter [4].

2. Let us try to apply the Wiener-Hopf technique to solution of (1.19). Together with (1.19) we introduce the following integro-differential equation into consideration:

$$g(t)-\frac{\lambda}{\pi}\int_{-\infty}^{\infty}\frac{g'(\tau)\,d\tau}{\tau-t}=b(t), \qquad -\infty<x<\infty \tag{2.1}$$

with the additional condition

$$g(0)=a \qquad (a=-p_0/h) \tag{2.2}$$

In this connection

$$b(t)=0 \quad \text{for } t\geqslant 0, \qquad b(t)=-\frac{\lambda}{\pi}\int_{-\infty}^{\infty}\frac{g'(\tau)\,d\tau}{\tau-t} \quad \text{for } t<0 \tag{2.3}$$

If the derivative of the function

$$g(t)=\begin{cases}v(t), & t\geqslant 0\\ 0, & t<0\end{cases} \tag{2.4}$$

where $v(t)$ is the solution of Eq. (1.19) with the required properties, was summable on the axis, then equations (1.19) and (2.1) would be equivalent among themselves. The derivative $g'(t)$, however, as follows from definition (2.4), represents the sum of an integrable function and a function of the type $\delta(t)$. Therefore, in order to obtain the required solution $v(t)$ of Eq. (1.19) from (2.1), the assumption of summability of the derivative of solution (2.1) should be relinquished and all necessary operations on this solution should be carried out in a purely formal manner.

In the following we take advantage of Fourier transformation. Let us agree to denote the functions by small letters and their Fourier transforms by the same capital letters. On the basis of (2.1)–(2.3) we have

$$\int_{-\infty}^{\infty} g'(\tau)\,e^{it\tau}\,d\tau=e^{it\tau}g(\tau)\big|_0^{\infty}-it\int_0^{\infty}g(\tau)\,e^{it\tau}\,d\tau=-a-itG(t) \tag{2.5}$$

In addition

$$\frac{1}{\pi}\int_{-\infty}^{\infty}e^{ixt}\,\frac{dx}{x-\tau}=i\ \operatorname{sig}\ te^{it\tau} \tag{2.6}$$

Performing the Fourier transformation on both parts of Eq. (2.1) and taking into account (2.5) and (2.6), we have

$$G(t)-i\lambda\,[a+itG(t)]\operatorname{sig}t=B(t) \qquad (-\infty<x<\infty)$$

By virtue of their definition functions $G(t)$ and $B(t)$ will be the limiting values of functions $G(t)$ and $B(z)$, which are actually holomorphic in S^+ and S^-, respectively. Therefore the previous equation can also be written as follows:

$$G^+(t)\,[1+\lambda|t|\,]=B^-(t)+i\lambda a\operatorname{sig}t \qquad (-\infty<t<\infty) \tag{2.7}$$

In this manner, for the determination of the desired Fourier transform $G(t)$ of solution (2.1), the problem of linear conjugation on the axis is obtained.

In the case of $G^+(t)$ in (2.7) we represent the coefficient in the form

$$1+\lambda|t|=\frac{1+\lambda|t|}{\sqrt{1+\lambda^2t^2}}\sqrt{1+i\lambda t}\,\sqrt{1-i\lambda t}$$

and bring into consideration the canonic solution $X(z)$ of the auxiliary problem of linear conjugation

$$X^+(t)=\frac{1+\lambda|t|}{\sqrt{1+\lambda^2t^2}}\,X^-(t) \qquad (-\infty<t<\infty) \tag{2.8}$$

The following function can be taken as $X(z)$:

$$X(z)=\exp\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\ln\frac{1+\lambda|\tau|}{\sqrt{1+\lambda^2\tau^2}}\frac{d\tau}{\tau-z}\right] \tag{2.9}$$

It is easy to verify that $X(z)$ satisfies the boundary condition (2.8), does not become zero anywhere, including the real axis, and for $|z|\to\infty$ in closed half-planes is:

$$X^{\pm}(\infty)=1 \tag{2.10}$$

Now let us represent the boundary condition (2.7) in the form

$$G^+(t)\sqrt{1-i\lambda t}X^+(t)=\frac{B^-(t)X^-(t)}{\sqrt{1+i\lambda t}}+\frac{i\lambda aX^-(t)\operatorname{sig} t}{\sqrt{1+i\lambda t}} \tag{2.11}$$

From this it follows directly that:

$$\sqrt{1-i\lambda z}X(z)G(z)=\frac{\lambda a}{2\pi}\int_{-\infty}^{\infty}\frac{X^-(t)\operatorname{sig} t}{\sqrt{1+i\lambda t}}\cdot\frac{dt}{t-z}\quad \text{for } z \text{ in } S^+$$

or on the basis of well-known Cauchy theorem

$$\sqrt{1-i\lambda z}X(z)G(z)=\frac{\lambda a}{\pi}\int_{0}^{\infty}\frac{X^-(t)}{\sqrt{1+i\lambda t}}\frac{dt}{t-z}\quad \text{for } z \text{ in } S^+ \tag{2.12}$$

Let us now determine the transformation of the Fourier function $g'(t)$. Let us designate it by $G_1(t)$ and note that values of this function represent boundary values $G_1^+(t)$ of the function holomorphic with respect to z in S^+. Having taken this into account we rewrite Eq. (2.5) in the equivalent form

$$G_1(z)=-a-izG(z) \tag{2.13}$$

Substituting into this $G(z)$ from (2.12), we find

$$G_1(z)=-a+\frac{\lambda az}{\sqrt{1-i\lambda z}X(z)}\frac{1}{\pi i}\int_{0}^{\infty}\frac{X^-(t)}{\sqrt{1+i\lambda t}}\frac{dt}{t-z} \tag{2.14}$$

Let us compute the limit of function $G_1(z)$ when the point moves out to infinity remaining all the time in the upper half-plane. To facilitate computation a substitution of variable z is performed in (2.14). The variable of integration is also changed

$$z=-1/\zeta,\qquad t=-1/\tau \tag{2.15}$$

Through transformation (2.15) the upper half-plane of plane z becomes the upper half-plane of the plane ζ, the real axis transforms into the real axis and the region of the point at infinity becomes the region of point $\zeta=0$. In this connection Eq. (2.14) transforms into the form

$$G_1^*(\zeta)+a=-\frac{\sqrt{\zeta}}{\sqrt{\theta+i\lambda}X^*(\zeta)}\frac{\lambda a}{\pi i}\int_{0}^{\infty}\frac{X^{*-}(\tau)}{\sqrt{\tau(\tau-i\lambda)}}\frac{d\tau}{\tau-\zeta} \tag{2.16}$$

$$G_1^*(\zeta)=G_1(z),\qquad X^*(\zeta)=X(z)$$

For evaluation of (2.16) in the case of small $|\zeta|$ we take advantage of the equation of Muskhelishvili which characterizes the behavior of an integral of the Cauchy type near the ends of the line of integration ([7], p. 78, Eq. (29.5)). We also use the limit equation (2.10). For points ζ in the vicinity of the origin of coordinates on the bisected plane the following equation is valid:

$$G_1^*(\zeta)+a=-a+o(1) \tag{2.17}$$

Returning from this to (2.14) we conclude that the function $G_0(z)$ holomorphic in S^+, which is determinable by the equation

$$G_0(z)=G_1(z)+2a=a+\frac{\lambda az}{\sqrt{1-i\lambda z}X(z)}\frac{1}{\pi i}\int_0^\infty \frac{X^-(t)}{\sqrt{1+i\lambda t}}\frac{dt}{t-z} \qquad (2.18)$$

vanishes at infinity. Its boundary values $G_0^+(t)$ will represent the Fourier transform of the desired function $v'(t)$

$$v'(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} G_0^+(\tau)\,e^{-it\tau}\,d\tau \qquad (2.19)$$

The preceding expression determines according to Eqs. (1.14) a piecewise holomorphic function $\varphi(z)$ which satisfies all requirements of the problem.

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The problem of internal semi-infinite stringer in an unbounded plate is solved in an analogous manner [3, 4].

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ON THE DEVELOPMENT OF CAVITIES IN VISCOUS BODIES

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The problem of the development of cavities in viscous bodies under infinite deformations is considered. A formation of the problem of the development of a cavity under the conditions of a stationary slow flow of a viscous Newtonian fluid is given in Sect. 1. An exact solution of the problem of broadening of the cavity from the initial viscous one is obtained in Sect. 2. The analysis is limited to the case of the plane problem.

1. Viscous body. Let us consider a viscous body subjected to Newton's law and occupying an infinite domain in the exterior of some contour L (the problem is considered a plane one). The interior of the contour L is some cavity whose shape is known only at